Now we are going to consider another enrichment of $\mathbf{C^*\text{-}Alg}$. This was introduced by Gennadi Kasparov and it is denoted by $\mathbf{KK}$. In a sense, it is a category of noncommutative topological motives, i.e. an universal cohomology theory of correspondences between noncommutative spaces.

For applications it is customary to consider C^*-algebras acted on by a fixed locally compact group G. The corresponding theory is denoted by KK_G and is called <u>G-equivariant KK-theory</u>.

We start from the classical topological context.

Definition. A G-space is a locally compact space X equipped with a continuous map

$$G \times X \to X \,, \quad (g,x) \mapsto gx$$

s.t. $g(g'x) = (gg')x$, $ex = x$.

Example. $H < G$ closed subgroup of a compact group G. $X := G/H$ is a compact Hausdorff space.

$$G \times G/H \to G/H \,, \quad (g, g'H) \mapsto (gg')H \,.$$

Definition. the <u>isotropy subgroup</u> (or <u>stabilizer</u>) of $x \in X$ $:\underline{=} \{ g \in G \mid gx = x \} =: G_x$.

Note that $\overline{G_x} = G_x$.

Exercise 19. Show that $\mathbb{CP}^n \cong U(n+1)/U(1)\times U(n)$.

where

$$U(1)\times U(n) \hookrightarrow \left\{ \left(\begin{array}{c|ccc} U(1) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U(n) & \\ 0 & & & \end{array} \right) \right\} \subset U(n+1).$$

Definition. By $\mathbf{CpctHaus}_G$ we denote the category of compact Hausdorff G-spaces with G-equivariant maps $f: X \to Y$ as morphisms (i.e. s.t. $f(gx) = gf(x)$).

Definition. For a closed subgroup $H < G$ one defines the pair of functors

$$\mathbf{CptHaus}_G \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} \mathbf{CptHaus}_H$$

$$\mathrm{Res}(G \times X \to X) = \left(\begin{tikzcd} H \times X \arrow[r] \arrow[rr, bend right] & G \times X \arrow[r] & X \end{tikzcd}\right)$$

$$\mathrm{Ind}(H \times X \to X) = G \times^H X := (G \times X)/H$$

where $\quad (g, x)h := (gh, h^{-1}x)$.

Exercise 20. Show that if a G-space admits a G-equivariant map $X \xrightarrow{f} G/H$, then $X = \mathrm{Ind}_H^G(Y)$ for some H-space Y.

Solution. If $X = G \times^H Y \xrightarrow{f} G/H$ then $Y = f^{-1}(eH)$, hence it must be so if X is induced from H.

Now, $G \times^H f^{-1}(eH) \longrightarrow X$, $[g, x] \longmapsto gx$ is G-equivariant if $g'[g,x] = [g'g, x]$ and admits a G-equivariant inverse

$$\begin{aligned} G \times^H f^{-1}(eH) &\longleftarrow X \\ (g, g^{-1}x) &\longleftarrow\!\shortmid\ x \end{aligned}$$

, where $f(x) = gH$. $\square$

Exercise 21. $(\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G)(X) = G/H \times X$, where the RHS is a diagonal G-space, i.e. $g'(gH, x) = (g'gH, g'x)$.

Solution. $G \times^H X \longrightarrow G/H \times X$, $[g, x] \longmapsto [gH, gx]$ is G-equivariant and has an inverse
$$[g, g^{-1}x] \longleftarrow\!\shortmid (gH, x). \quad \blacksquare$$

Theorem. [Palais] Let G be a Lie group, acting properly on X. Then $\forall_{x \in X}\ \exists$ G-invariant open neighborhood U of x and a G-map $U \longrightarrow G/G_x$, i.e. $U = \mathrm{Ind}_{G_x}^G(V)$. $\blacksquare$

Corollary. If a Lie group G acts freely on X, then locally X looks like $G \times Y$ for some Y.

Fact. Assume X is a smooth manifold with G a Lie group acting smoothly on X. Let Gx be an orbit of $x \in X$ and N be its normal bundle. Then the vector space N_x is a representation of G_x and the tubular neighborhood U of the orbit can be taken as $U \cong G \times^{G_x} N_x$.

Definition. A __vector bundle__ over a (compact Hausdorff) space is a space E equipped with

a) a map $\pi : E \longrightarrow X$

b) a structure of a finite-dimensional vector space on each fibre $E_x = \pi^{-1}(x)$

c) $\forall_{x \in X} \exists_{\underset{\text{open}}{U \ni x}}$ $\pi^{-1}(U) \cong U \times \mathbb{C}^n$ linear on each fibre.

Definition. A vector bundle $E \xrightarrow{\pi} X$ is __G-equivariant__ if π is a G-equivariant map of G-spaces, and the action $E_x \longrightarrow E_{gx}$ is linear.

Example. Let X be a G-space and $\rho : G \to GL(V)$ be a finite-dimensional representation. Then $X \times V$ with a diagonal G-action is a G-equivariant vector bundle on X with respect to $\pi : X \times V \to X$ being the canonical projection.

Example. If G acts smoothly on a compact manifold X then $TX \to X$ is a G-equivariant vector bundle.

Example. Let $X \to Y$ be an embedding of G-manifolds. Then the normal bundle of X in Y is a G-equivariant vector bundle.

Example. If $H < G$ is a closed subgroup and $\rho : H \to GL(V)$ is a finite-dimensional representation, then $G \times^H V \longrightarrow G/H$, $[g, v] \longmapsto gH$ is a G-equivariant vector bundle. This is called the <u>induced vector bundle</u>.

Note that if ρ extends to a representation of G then this bundle is trivial: $G \times^H V = G/_H \times V$.

Example. Let $f : X \to Y$ be a G-equivariant map of G-spaces and $F \to Y$ is a G-equivariant vector bundle. Then $E = f^* F = X \times_Y F \longrightarrow X$ is a G-equivariant v.b.

This is called the <u>pull-back</u> of an equivariant vector bundle.

Theorem. If $f_0, f_1 : X \to Y$ are G-equivariant maps which are homotopic, then $f_0^* F \cong f_1^* F$ for any G-equivariant vector bundle F on Y. $\square$

Operations on G-equivariant vector bundles.

$$E_1 \oplus E_2 := E_1 \times_X E_2 = \{ (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2) \}$$

$$E_1 \otimes E_2 := \coprod_{x \in X} E_{1x} \otimes E_{2x}$$

Definition. $\mathrm{Vect}_G(X) :=$ abelian monoid of isomorphism classes of G-vector bundles over X with respect to the direct sum.

The twisting by a representation $\rho : G \to GL(V)$ is $E \otimes (X \times V)$.

Definition. The G-equivariant K-theory $K_G^0(X)$ of X is the Grothendieck group of the monoid $\mathrm{Vect}_G(X)$. It is a ring over $R(G)$, the representation ring of G. Note that $K_G^0(*) = R(G)$.

Remark. If G is trivial, then $R(G) = \mathbb{Z}$ and $K^0(X) := K^0_G(X)$ is simply an abelian group.

If X is a trivial G-space, then $K^0_G(X) \cong K^0(X) \otimes_{\mathbb{Z}} R(G)$.

Theorem. [Serre-Swann] Let X be compact Hausdorff. Then $E \longmapsto \Gamma(E)$, the space of sections induces an equivalence of categories

$$\mathbf{Vect}(X) \xrightarrow{\sim} \mathbf{f.g.p\,Mod}(C(X))$$

vector bundles on X

finitely generated projective modules over $C(X)$

Proof.

Lemma. If X is paracompact (i.e. compact), any exact sequence of vector bundles over X

$$0 \to E' \xrightarrow{\iota} E \xrightarrow{\pi} E'' \to 0$$

splits. In particular, the canonical sequence

$$0 \to E' \xrightarrow{\iota} E \xrightarrow{\pi} E/E' \to 0,$$

for a subbundle E' of E, splits.

Proof. At each $x \in X$ the short exact sequence of vector spaces $0 \to E'_x \to E_x \to E''_x \to 0$ splits.

There is an open neighborhood U of x s.t. π is represented by a matrix function $p : U \to \mathrm{Vect}(\mathbb{C}^j, \mathbb{C}^n)$ where $\mathrm{rank}(p(x)) = r$. Since the set of matrices of rank r is open in $\mathrm{Vect}(\mathbb{C}^j, \mathbb{C}^r)$, there is an open neighborhood V of x such that $\mathrm{rank}\, p|_V = r$ at every point of V so the sequence

$$\begin{tikzcd} 0 \arrow[r] & E'|_V \arrow[r] & E|_V \arrow[r] & E''|_V \arrow[l, bend right, dotted, "\sigma_V"'] \arrow[r] & 0 \end{tikzcd}$$

splits. We may assume that such V's form a locally finite covering of X. Let (φ_V) be a partition of unity subordinate to this covering.

Set now
$$\sigma := \sum_V \varphi_V \sigma_V .$$
Then
$$\pi\sigma = \sum_V \varphi_V \pi|_V \sigma_V = \sum_V \varphi_V \, Id_{E''|_V} = \Big(\sum_V \varphi_V\Big) Id_{E''}$$
$$= Id_{E''}$$
hence the short exact sequence splits. $\blacksquare$

Lemma. Let E be a vector bundle over a paracompact space X with a finite open trivializing cover (i.e. X compact). Then there exists a vector bundle E' s.t. $E'\oplus E \to X$ is trivial.

Proof. (U_i) finite open covering of X trivializing E, $s_{ij} \in \Gamma(U_i, E)$ linearly independent sections, (φ_i) partition of unity subordinate to (U_i). Define $\sigma_{ij} := \varphi_i s_{ij}$. Then σ_{ij} span E_x at every $x \in X$, so we have an epimorphism $\pi : X \times \mathbb{C}^N \longrightarrow E$, hence we have a short exact sequence of v. bundles

$$0 \to E' \xrightarrow{\iota} X \times \mathbb{C}^N \xrightarrow{\pi} E \to 0$$

where $E' = \ker \pi$ is a vector bundle.

By the previous lemma it splits, hence $\exists E'$ s.t.

$$E' \oplus E \cong X \times \mathbb{C}^N. \quad \square$$

Lemma. Each $C(X)$-linear map $\Gamma(E) \to \Gamma(F)$ is of the form $\Gamma(\alpha)$ for some vector bundle map $\alpha : E \to F$, i.e. $\mathrm{Vect}_X(E, F) \to \mathrm{Mod}_{C(X)}(\Gamma(E), \Gamma(F))$ is surjective.

Proof. The bundle maps α form the module of sections of $\mathcal{H}om(E, F) = E^* \otimes F \longrightarrow X$,

$\Rightarrow \alpha \in \Gamma(E^* \otimes F)$. On the other hand, $C(X)$-linear maps $\Gamma(E) \to \Gamma(F)$ belongs to

$$\mathrm{Mod}_{C(X)}(\Gamma(E), \Gamma(F)) = \Gamma(E)^\vee \otimes_{C(X)} \Gamma(F)$$

The latter equality follows from the fact that $\Gamma(E^*) \cong \Gamma(E)^\vee$ and the fact that for v. bundles E, F
$\Gamma(E \oplus F) \cong \Gamma(E) \otimes_{C(X)} \Gamma(F)$.

Both facts are obvious for trivial E and F.

In general, we can complement E and F to trivial $E' \oplus E$ and $F' \oplus F$, and from a commutative diagram

$$
\begin{tikzcd}
\Gamma(E)\otimes_{C(X)}\Gamma(F) \arrow[r, "\alpha"] \arrow[d, "\Gamma(\sigma_E)\otimes\Gamma(\sigma_F)"'] & \Gamma(E\otimes F) \arrow[d, "\Gamma(\sigma)"] \\
\Gamma(E'\oplus E)\otimes_{C(X)}\Gamma(F'\oplus F) \arrow[r, "\cong"] \arrow[d, "\Gamma(\pi_E)\otimes\Gamma(\pi_F)"'] & \Gamma\big((E'\oplus E)\otimes(F'\oplus F)\big) \arrow[d, "\Gamma(\pi)"] \\
\Gamma(E)\otimes_{C(X)}\Gamma(F) \arrow[r, "\alpha"] & \Gamma(E\otimes F)
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "\iota"] & E'\oplus E \arrow[r, shift right, "\pi"'] & E \arrow[l, shift right, dashed, "\sigma_E"'] \arrow[r] & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[r] & (E'\otimes F')\oplus(E'\otimes F)\oplus(E\otimes F') \arrow[r] & (E'\oplus E)\otimes(F'\oplus F) \arrow[r, shift right, "\pi"'] & E\otimes F \arrow[l, shift right, dashed, "\sigma"'] \arrow[r] & 0
\end{tikzcd}
$$

are canonical splittings.

The upper square yields that α is mono, the bottom square yields that α is epi, hence in general α is an isomorphism of $C(X)$-modules.

Corollary. The functor of global sections $E \longmapsto \Gamma(E)$ is faithful, full and exact.

Proof. "Faithful" means that $\Gamma(\alpha) = \Gamma(\beta)$ implies $\alpha = \beta$, which is clear from the definition of Γ. "Full" means that $\mathrm{Vect}_X(E, F) \to \mathrm{Mod}_{C(X)}(\Gamma(E), \Gamma(F))$ is surjective, which we have just established.

"Exact" means that Γ preserves short exact sequences. But all these split, so under Γ they are (split) exact. $\square$

Lemma. If X is compact Hausdorff, the $C(X)$-modules $\Gamma(E)$ are finitely generated and projective.

Proof. $\exists E'$ such that $E' \oplus E$ is trivial so we have a split short exact sequence

$$0 \to E' \to X \times \mathbb{C}^{N} \xrightarrow{\;E\;} E \to 0$$

which induces a split short exact sequence

$$0 \longrightarrow \Gamma(E') \longrightarrow C(X)^N \underset{\pi}{\overset{\xleftarrow{\sigma}}{\longrightarrow}} \Gamma(E) \longrightarrow 0$$

so $\Gamma(E)$ is a direct summand in the free $C(X)$-module $C(X)^N$. $\square$

Remark. $\sigma\pi$ can be regarded as an idempotent matrix $e = e^2 \in M_N(C(X))$ with entries in $C(X)$ and $\Gamma(E) \cong e\, C(X)^N$ as right $C(X)$-modules.